

Improved bounds for the RIP of Subsampled Circulant Matrices

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Abstract. In this paper, we study the restricted isometry property of partial random circulant matrices. For a bounded subgaussian generator with independent entries, we prove that the partial random circulant matrices satisfy s -order RIP with high probability if one chooses $m \gtrsim s \log^2 s \log n$ rows randomly where n is the vector length. This improves the previously known bound $m \gtrsim s \log^2 s \log^2 n$.

Key words and phrases : Restricted isometry property, Partial random circulant matrices, Subgaussian random vectors

2010 AMS Mathematics Subject Classification — Primary 15B52, 15B05, Secondary 60G15, 60F10

1. Introduction

1.1. Compressed sensing. The aim of compressed sensing [6, 9, 13, 22] is to recover s -sparse signals $\mathbf{x} \in \mathbb{C}^n$ from the linear measurements $\mathbf{y} = \mathbf{A}\mathbf{x}$ and the compressed sensing matrix $\mathbf{A} \in \mathbb{C}^{m \times n}$ with $m < n$. Here, we say $\mathbf{x} \in \mathbb{C}^n$ is s -sparse if $\|\mathbf{x}\|_0 \leq s$ where $\|\mathbf{x}\|_0$ denotes the number of nonzero entries of \mathbf{x} .

*Zhiqiang Xu was supported by NSFC grant (91630203, 11688101), Beijing Natural Science Foundation (Z180002).

A naive approach for reconstructing \mathbf{x} is to solve the following ℓ_0 -minimization problem

$$(P_0) \quad \min_{\mathbf{z}} \|\mathbf{z}\|_0 \quad \text{subject to} \quad \mathbf{A}\mathbf{z} = \mathbf{y}.$$

However, the ℓ_0 -minimization problem is NP-hard [20] which is not tractable. A natural approach is to relax ℓ_0 -minimization to ℓ_1 -minimization [6, 8, 9, 13], i.e.,

$$(P_1) \quad \min_{\mathbf{z}} \|\mathbf{z}\|_1 \quad \text{subject to} \quad \mathbf{A}\mathbf{z} = \mathbf{y}.$$

The (P_1) is a convex optimization problem which can be solved efficiently. To guarantee the reconstruction of the sparse signals \mathbf{x} by (P_1) , it suffices to require the matrix \mathbf{A} satisfying restricted isometry property (RIP). For a matrix $\mathbf{A} \in \mathbb{R}^{m \times n}$ and an integer $s \in [1, n]$, we say \mathbf{A} satisfies s -order RIP with constant $\delta_s \in [0, 1)$ if

$$(1 - \delta_s)\|\mathbf{x}\|_2^2 \leq \|\mathbf{A}\mathbf{x}\|_2^2 \leq (1 + \delta_s)\|\mathbf{x}\|_2^2 \quad (1)$$

holds for all s -sparse signals \mathbf{x} . It has been proved that if the matrix \mathbf{A} satisfies $t \cdot s$ -order RIP with $\delta_{ts} < \sqrt{1 - \frac{1}{t}}$ for $t > 4/3$, then every s -sparse vector \mathbf{x} can be reconstructed by solving (P_1) [7, 5]. Moreover, RIP is also employed to study the performance of greedy algorithms for the recovery of sparse signals, such as OMP [26], OMMP [25], CoSaMP [11, 21], iterative hard thresholding [3] and hard thresholding pursuit [10]. Hence, one is interested in constructing RIP matrices $\mathbf{A} \in \mathbb{C}^{m \times n}$ with m being as small as possible. A popular method for constructing RIP matrices is to use random matrices. For example, Gaussian random matrix $\frac{1}{\sqrt{m}}\mathbf{A} \in \mathbb{R}^{m \times n}$ satisfies s -order RIP with high probability provided that $m \geq Cs \log(n/s)$, where the entries of \mathbf{A} are independent standard normal random variables. From the Gelfand widths, the lower bound $Cs \log(n/s)$ is optimal up to a constant [12]. Other random matrices which can achieve this bound include Bernoulli matrices and subgaussian matrices [18, 22]. In practical applications, one prefers structured random matrices since they can make the recovery algorithms more efficient. An important example of structured random matrices are partial Fourier matrices whose rows are selected randomly from the discrete Fourier matrix. It has been shown that partial Fourier matrices satisfy s -order RIP with high probability provided that $m \geq Cs \log^2 s \log n$, (see [15, 22, 24, 4]).

The aim of this paper is to study the RIP of the partial random circulant matrices. Compared to Bernoulli or Gaussian matrices, the partial random circulant matrices have the advantage that they reduce the generation of only n independent random variables instead of n^2 . More importantly, they admit fast matrix-vector multiplication and arise naturally in certain applications such as in radar, aperture imaging [14, 23] as well as MR imaging [17]. Hence, they attract much attention.

1.2. **Related work.** Assume that $\xi = (\xi_1, \dots, \xi_n) \in \mathbb{C}^n$. We define the circulant matrix generated by ξ as

$$\mathbf{A}_\xi := \begin{pmatrix} \xi_1 & \xi_n & \cdots & \xi_2 \\ \xi_2 & \xi_1 & \cdots & \xi_3 \\ \vdots & \vdots & \ddots & \vdots \\ \xi_n & \xi_{n-1} & \cdots & \xi_1 \end{pmatrix} \in \mathbb{C}^{n \times n}.$$

For a multiset Ω in $\{1, \dots, n\}$ with cardinality m , let $P_\Omega : \mathbb{C}^n \rightarrow \mathbb{C}^m$ denote the projection operator that restricts a vector $\mathbf{x} \in \mathbb{C}^n$ to its entries in Ω . Then the corresponding partial circulant matrix generated by ξ is defined as

$$\Phi_\xi = \frac{1}{\sqrt{m}} P_\Omega \mathbf{A}_\xi \in \mathbb{C}^{m \times n}.$$

We next introduce another family of structured random matrices. For $\xi = (\xi_1, \dots, \xi_{2n-1}) \in \mathbb{C}^{2n-1}$, the Toeplitz matrix T_ξ is defined as

$$T_\xi = \begin{pmatrix} \xi_n & \xi_{n-1} & \cdots & \xi_1 \\ \xi_{n+1} & \xi_n & \cdots & \xi_2 \\ \vdots & \vdots & \ddots & \vdots \\ \xi_{2n-1} & \xi_{2n-2} & \cdots & \xi_n \end{pmatrix}$$

and the Hankel matrix is defined as $H_\xi = T_\xi J_n$ where $J_n = [\mathbf{e}_n, \dots, \mathbf{e}_1]$ and $\mathbf{e}_j, j = 1, \dots, n$ are the standard orthogonal vectors. For a multiset Ω of $\{1, \dots, n\}$ with cardinality m , the corresponding partial Toeplitz matrix

$$\Psi_\xi = \frac{1}{\sqrt{m}} P_\Omega T_\xi \in \mathbb{C}^{m \times n}. \quad (2)$$

Similarly, we can define the partial Hankel matrices. A simple observation is that $\frac{1}{\sqrt{m}} P_\Omega T_\xi$ satisfies s -order RIP with the constant δ_s if and only if $\frac{1}{\sqrt{m}} P_\Omega H_\xi$ satisfies the same property. As we will see later, a Toeplitz matrix can be embedded in a circulant matrix of twice the dimension. Hence, many RIP results for partial random circulant matrices can be extended to the partial random Toeplitz matrices as well as partial random Hankel matrices.

In [23], under the setup in which both Ω as well as the generating vector ξ are chosen at random, Romberg proves that $m \gtrsim s \log^6 n$ measurements are sufficient to guarantee $\Phi_\xi = \frac{1}{\sqrt{m}} P_\Omega \mathbf{A}_\xi \in \mathbb{C}^{m \times n}$ satisfying RIP. For an arbitrary fixed selection of $\Omega \subset \{1, \dots, n\}$, the first theoretical results are established in [2] with showing that $m \times n$ partial random circulant matrices satisfy s -order RIP with high probability provided $m \gtrsim s^3 \log n$. This is then improved by Bajwa et al. [1, 14] to $m \gtrsim s^2 \log n$ for general matrices whose entries were drawn from bounded or Gaussian distribution. Later, Rauhut, Romberg and Tropp prove that $m \gtrsim s^{3/2} \log^{3/2} n$ measurements are enough. And the result is then improved by Krahmer, Mendelson and Rauhut [16] to $O(s \log^2 s \log^2 n)$. Recently, Mendelson, Rauhut and Ward [19] prove that $O(s \log^2 s \log(\log s) \log n)$ measurements for partial random circulant matrix with random sampling set Ω

and Gaussian random generator are sufficient to recover all s -sparse vectors with high probability via ℓ_1 -minimization, but it does not establish any type of RIP.

1.3. Main results. In this paper, we study the RIP of the partial random circulant matrices with bounded or Gaussian entries, where the subsampling locations Ω are selected at random from $\{1, \dots, n\}$. We show that, if $m \gtrsim s \log^2 s \log n$ then $\Phi_\xi = \frac{1}{\sqrt{m}} P_\Omega \mathbf{A}_\xi$ satisfies s -order RIP with high probability where ξ is a bounded random vector. This improves upon the best previously known bound $O(s \log^2 s \log^2 n)$ [16]. Some suitable bounded random vectors ξ include uniform distribution $\xi_i \sim \text{U}(-\sqrt{3}, \sqrt{3})$ and Rademacher vector with $\mathbb{P}(\xi_i = \pm 1) = 1/2$.

Theorem 1.1. *Let $\xi = (\xi_1, \dots, \xi_n) \in \mathbb{C}^n$ be a random vector whose entries are i.i.d. realizations of bounded zero-mean random variables satisfying $\mathbb{E}\xi_j^2 = 1$ and $|\xi_j| \leq c$ for some $c \geq 1$. Suppose that $\delta > 0$ is a sufficient small constant and $m \gtrsim \log^2(1/\delta) \cdot \delta^{-2} s \log^2(s/\delta) \log n$. Let Ω be a multiset of m uniform and independent random elements of $\{1, \dots, n\}$ and $\Phi_\xi = \frac{1}{\sqrt{m}} P_\Omega \mathbf{A}_\xi \in \mathbb{C}^{m \times n}$ be a partial random circulant matrix generated by ξ and Ω . If $s \lesssim n/(\log^4 n)$, then the matrix Φ_ξ satisfies the restricted isometry property with order s and constant $\delta_s \leq \delta$ with probability at least $1 - 2^{-C \log n \log(s/\delta)} - n^{-\log n \log^2 s}$.*

In the practical applications, one usually requires that $s \ll n$. Hence, the assumption of $s \lesssim n/\log^4 n$ in Theorem 1.1 is enough for the practical applications. The idea for proving Theorem 1.1 is to use result from [15] (see Theorem 2.4 in Section 2), which shows the connection between $\frac{1}{n} \|M\mathbf{x}\|_2^2$ and $\frac{1}{m} \|P_\Omega M\mathbf{x}\|_2^2$ for any matrix $M \in \mathbb{C}^{n \times n}$ (see (3) for detail). In [15], Haviv and Regev consider the case where M is a DFT matrix for which $\frac{1}{n} \|M\mathbf{x}\|_2^2 = \|\mathbf{x}\|_2^2$. A particular case of Theorem 2.2 from [16] shows that the circulant matrix \mathbf{A}_ξ satisfies $\frac{1}{n} \|\mathbf{A}_\xi \mathbf{x}\|^2 \approx \|\mathbf{x}\|^2$ provided $s \lesssim n/\log^4 n$ (see Corollary 2.3). Combining them, we can arrive at Theorem 1.1. The proof just combines the known results, but Theorem 1.1 definitely goes far beyond the current state-of-the-art.

Remark 1.2. *We consider the case with removing the assumption of ξ_j being a bounded random variable. Assume that $\xi \in \mathbb{C}^n$ is a random vector with independent, mean 0 and variance 1, L -subgaussian entries. We can obtain that $\mathbb{P}(|\xi_j| > L\sqrt{2 \log n}) \leq 2/n^2$ for any ξ_j . Then*

$$\mathbb{P}\left(\max_j |\xi_j| \leq L\sqrt{2 \log n}\right) \geq 1 - 2/n,$$

which implies that $|\xi_j| \leq L\sqrt{2 \log n}$ for all $j = 1, \dots, n$ with probability at least $1 - 2/n$. Since the bound $L\sqrt{2 \log n}$ is not a constant, it leads to slightly larger samples $m \gtrsim s \log^2(s \log n) \log^2 n$ for the partial random circulant matrix Φ_ξ to satisfy s -order RIP. Nevertheless, the result can match the best bound $m \gtrsim s \log^2(s) \log^2(n)$ which is obtained in [16] provided $s \geq \log n$.

Remark 1.3. Note that a Toeplitz matrix can be embedded in a circulant matrix of twice the dimension. Then we can show that Toeplitz matrix satisfies $\frac{1}{n}\|T_\xi \mathbf{x}\|^2 \approx \|\mathbf{x}\|^2$ provided $s \lesssim n/\log^4 n$. Hence, our result can be extended to the partial random Toeplitz matrices.

2. Preliminaries

We first introduce the definition of L -subgaussian random vectors which include the Rademacher vectors as well as the standard Gaussian vectors as special cases. To state conveniently, we use \mathcal{S}^{n-1} to denote the unit sphere in \mathbb{C}^n .

Definition 2.1 (L -subgaussian). A mean 0 random vector $X \in \mathbb{C}^n$ is called isotropic if for every $\theta \in \mathcal{S}^{n-1}$, $\mathbb{E}|\langle X, \theta \rangle| = 1$. A random vector X is called L -subgaussian if it is isotropic and $\mathbb{P}(|\langle X, \theta \rangle| \geq t) \leq 2\exp(-t^2/2L^2)$ for every $\theta \in \mathcal{S}^{n-1}$ and any $t \geq 0$.

We introduce some known results which are useful in our analysis.

Theorem 2.2. ([16, Theorem 4.1]) For any fixed subset $\Omega \subset \{1, \dots, n\}$ and a random vector $\xi \in \mathbb{C}^n$ with independent, mean 0 and variance 1, L -subgaussian entries. Let $\Phi_\xi = \frac{1}{\sqrt{m}}P_\Omega \mathbf{A}_\xi \in \mathbb{R}^{m \times n}$ be a partial random circulant matrix generated by ξ and Ω . If

$$m \geq c\delta^{-2}s \log^2 s \log^2 n,$$

then with probability at least $1 - n^{-\log n \log^2 s}$, the matrix Φ_ξ satisfies the restricted isometry property with constant $\delta_s \leq \delta$. The constant $c > 0$ is universal.

If we take $\Omega = \{1, \dots, n\}$, then we have the following corollary:

Corollary 2.3. Suppose that $\xi \in \mathbb{C}^n$ is a random vector with independent, mean 0 and variance 1, L -subgaussian entries. Suppose that $s \lesssim n/\log^4 n$. Then

$$(1 - \delta)\|\mathbf{x}\|^2 \leq \frac{1}{n}\|\mathbf{A}_\xi \mathbf{x}\|^2 \leq (1 + \delta)\|\mathbf{x}\|^2 \quad \text{for all } s\text{-sparse } \mathbf{x} \in \mathbb{C}^n$$

holds with probability at least $1 - n^{-\log n \log^2 s}$.

The above corollary shows that we can obtain $(1 - \delta)\|\mathbf{x}\|^2 \leq \frac{1}{n}\|\mathbf{A}_\xi \mathbf{x}\|^2 \leq (1 + \delta)\|\mathbf{x}\|^2$ for all s -sparse vector \mathbf{x} provided $s \lesssim n/\log^4 n$.

We next introduce the main result in [15] which also plays an important role in our analysis.

Theorem 2.4. ([15, Theorem 4.1]) For a sufficiently large n , a matrix $M \in \mathbb{C}^{n \times n}$, and sufficiently small $\varepsilon, \eta > 0$, the following holds. For some $m = O(\log^2(1/\varepsilon) \cdot \varepsilon^{-1} \eta^{-1} \log n \cdot \log^2(1/\eta))$, let Ω be a multiset of m uniform and independent random elements of $\{1, \dots, n\}$. Then, with probability $1 - 2^{-\Omega(\log n \cdot \log(1/\eta))}$, it holds that for every $\mathbf{x} \in \mathbb{C}^n$,

$$\frac{1}{n}(1 - \varepsilon)\|M\mathbf{x}\|_2^2 - \eta\|\mathbf{x}\|_1^2 \|M\|_\infty^2 \leq \frac{1}{m} \sum_{j \in \Omega} |(M\mathbf{x})_j|^2 \leq \frac{1}{n}(1 + \varepsilon)\|M\mathbf{x}\|_2^2 + \eta\|\mathbf{x}\|_1^2 \|M\|_\infty^2, \quad (3)$$

where $\|M\|_\infty := \max_{i,j} |M_{i,j}|$.

3. Proofs of Theorem 1.1

Before giving the proof of Theorem 1.1, we next introduce a proposition which shows that $\frac{1}{n}\|\mathbf{A}_\xi \mathbf{x}\|^2 \approx \|\mathbf{x}\|^2$ does not hold for all $\mathbf{x} \in \mathbb{C}^n$ with high probability. Hence, to guarantee $\frac{1}{n}\|\mathbf{A}_\xi \mathbf{x}\|^2 \approx \|\mathbf{x}\|^2$ we need to require \mathbf{x} lies in some subset in \mathbb{C}^n . For example, in Theorem 1.1, we require $s = \|\mathbf{x}\|_0 \lesssim n/\log^4 n$. This also shows the essential difference between \mathbf{A}_ξ and Fourier matrices.

To state conveniently, we say a vector is a Gaussian random vector if the entries are i.i.d. standard Gaussian random variables.

Proposition 3.1. *Let $\xi = (\xi_1, \dots, \xi_n) \in \mathbb{R}^n$ be a Rademacher vector or Gaussian random vector and $\mathbf{A}_\xi \in \mathbb{R}^{n \times n}$ be the random circulant matrix generated by ξ . Then for any fixed $\epsilon > 0$, there exists a vector $\mathbf{x} \in \mathbb{R}^n$ and a positive constant p_0 so that*

$$\mathbb{P}\left(\frac{1}{n}\|\mathbf{A}_\xi \mathbf{x}\|_2^2 < \epsilon\right) \geq p_0 \quad \text{when } n \text{ is large enough.}$$

Proof. Setting $\mathbf{x} = \frac{1}{\sqrt{n}}(1, \dots, 1)^\top$, we have

$$\frac{1}{n}\|\mathbf{A}_\xi \mathbf{x}\|_2^2 = \left(\frac{\xi_1 + \dots + \xi_n}{\sqrt{n}}\right)^2 := \zeta_n^2.$$

For Gaussian random vector ξ , a simple observation is that $\zeta_n \sim \mathcal{N}(0, 1)$. It implies that

$$\mathbb{P}\left(\frac{1}{n}\|\mathbf{A}_\xi \mathbf{x}\|_2^2 < \epsilon\right) = \mathbb{P}(|\zeta_n| < \sqrt{\epsilon}) = 2(1 - \Phi(\sqrt{\epsilon})) := p_0,$$

where $\Phi(x)$ is the cumulative distribution function of standard Gaussian random variable. For Rademacher vector ξ , note that $\{\xi_i\}$ is a i.i.d. random variable sequence with $\mathbb{E}(\xi_i) = 0$ and $\mathbb{E}(\xi_i^2) = 1$. Recall that

$$\mathbb{P}\left(\frac{1}{n}\|\mathbf{A}_\xi \mathbf{x}\|_2^2 < \epsilon\right) = \mathbb{P}(|\zeta_n| < \sqrt{\epsilon}).$$

Then by Central Limit Theorem one finds that $\mathbb{P}(|\zeta_n| < \sqrt{\epsilon})$ tends to $2(1 - \Phi(\sqrt{\epsilon}))$ with n tending to infinity. Hence, we arrive at conclusion. \square

We next give the proof of Theorem 1.1.

Proof of Theorem 1.1. Through out this proof, we assume that $\|\mathbf{x}\|_0 \leq s$. Recall that $\Phi_\xi = \frac{1}{\sqrt{m}}P_\Omega \mathbf{A}_\xi$. From Corollary 2.3, we obtain that

$$(1 - \delta/4)\|\mathbf{x}\|_2^2 \leq \frac{1}{n}\|\mathbf{A}_\xi \mathbf{x}\|_2^2 \leq (1 + \delta/4)\|\mathbf{x}\|_2^2, \quad \text{for all } s\text{-sparse vectors } \mathbf{x} \quad (4)$$

holds with probability at least $1 - n^{-\log n \log^2 s}$ provided $s \lesssim \delta^2 n / (\log^2 s \log^2 n)$. Cauchy-Schwarz inequality implies that $\|\mathbf{x}\|_1 \leq \sqrt{s} \|\mathbf{x}\|_2$. Observe that

$$\|\Phi_\xi \mathbf{x}\|_2^2 = \frac{1}{m} \sum_{j \in \Omega} |(\mathbf{A}_\xi \mathbf{x})_j|^2.$$

For the given multiset Ω , choosing $\eta \geq \delta / (4c^2 s)$ in Theorem 2.4 for a fixed circulant matrix \mathbf{A}_ξ with $\|\mathbf{A}_\xi\|_\infty \leq c$, then with probability at least $1 - 2^{-C \log n \log(s/\delta)}$ we have

$$\frac{1}{n} (1 - \delta/4) \|\mathbf{A}_\xi \mathbf{x}\|_2^2 - \delta/4 \|\mathbf{x}\|_2^2 \leq \|\Phi_\xi \mathbf{x}\|_2^2 \leq \frac{1}{n} (1 + \delta/4) \|\mathbf{A}_\xi \mathbf{x}\|_2^2 + \delta/4 \|\mathbf{x}\|_2^2 \quad (5)$$

provided $m = O(\log^2(1/\delta) \cdot \delta^{-2} s \log^2(s/\delta) \log n)$.

Combining (4) and (5), we obtain that

$$(1 - \delta) \|\mathbf{x}\|_2^2 \leq \|\Phi_\xi \mathbf{x}\|_2^2 \leq (1 + \delta) \|\mathbf{x}\|_2^2$$

holds with probability at least $1 - 2^{-C \log n \log(s/\delta)} - n^{-\log n \log^2 s}$, which arrives at the conclusion. \square

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